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# BRAIDS AND NIELSEN-THURSTON TYPES OF AUTOMORPHISMS OF PUNCTURED SURFACES


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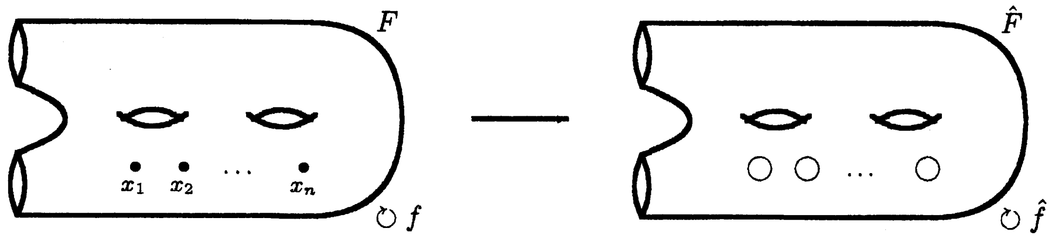
ABSTRACT. Let $F$ be a compact, orientable surface with negative Euler characteristic, and let $x_1, \cdots, x_n$ be $n$ fixed but arbitrarily chosen points on int$F$; each $x_i$ has a (small) diskal neighborhood $D_i \subset F$. Denote by $\mathcal{S}_n(F)$ a subgroup of Diff$(F)$ consisting of "sliding" maps $f$ each of which satisfies (1) $f(\{x_1, \dots, x_n\}) = \{x_1, \dots, x_n\}$, $f(D_1 \cup \cdots \cup D_n) = D_1 \cup \cdots \cup D_n$ and (2) $f$ is isotopic to the identity map on $F$. Then by restricting such automorphisms to $\hat{F} = F - \mathrm{int}(D_1 \cup \cdots \cup D_n)$, we have automorphisms $\hat{f} : \hat{F} \to \hat{F}$, which form a subgroup $\mathcal{S}_n(\hat{F})$ of Diff$(\hat{F})$. We give a Nielsen-Thurston classification of elements of $\mathcal{S}_n(\hat{F})$ using braids in $F \times I$ which characterize the elements of $\mathcal{S}_n(\hat{F})$.


## 1. INTRODUCTION

1.1. **記号と設定.** 本稿では、向き付け可能曲面の向きを保つ自己微分同相写像を**同型写像**(automorphism) ということとし、$F$ で負のオイラー標数を持つコンパクト、向き付け可能曲面を表す。$F$ の内部に $n$ 個の点 $x_1, \cdots, x_n$ を任意に選び、各 $x_i$ の (十分小さな) 近傍 $D_i \subset F$ をとっておく。$F - \mathrm{int}(D_1 \cup \cdots \cup D_n)$ を $\hat{F}$ で表し、同型写像 $f$ の $\hat{F}$ への制限を $\hat{f}$ で表す。




$F$ の同型写像 $f$ に対し、次の条件を考える:
(1) $f(\{x_1, \cdots, x_n\}) = \{x_1, \cdots, x_n\}$ かつ $f(D_1 \cup \cdots \cup D_n) = D_1 \cup \cdots \cup D_n$。
(2) $f$ は恒等写像と $F$ 上でイソトピック。
これらの条件を満たす同型写像全体からなる、$F$ の微分同相写像群 (Diff$(F)$ で表す) の部分群を $\mathcal{S}_n(F)$ で表すことにする。さらに、$\mathcal{S}_n(\hat{F})$ を $\{\hat{f} \mid f \in \mathcal{S}_n(F)\}$ で定まる Diff$(\hat{F})$ の部分群とする。

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1.2. **背景.** 空でない境界を持つコンパクト、向き付け可能曲面の同型写像について、その何回かの合成が恒等写像に一致するとき、**周期的**(periodic) であるといい、それが本質的 1 次元部分多様体を不変にするとき、**可約**(reducible) であるという。ここで、本質的 1 次元部分多様体 (essential 1-submanifold) とは、互いに交わらない単純閉曲線の族で、各々が境界平行でも 1 点にホモトピックでもなく、どの 2 本も互いにホモトピックでないもののことである。

考えている曲面のオイラー標数が負のとき、同型写像が周期的なものにも可約なものにもイソトピックでない為の必要十分条件は、それが**擬アノソフ**(pseudo-Anosov) なものにイソトピックであることが知られている (擬アノソフ同型写像の正確な定義など、詳しくは [11]、[6, Exposé 11, see also p.286]、[2] を参照)。従って、任意の同型写像は上記の 3 種類のもののどれかにイソトピックとなる。これらの型を、**ニールセン・サーストン型**と呼ぶことにする。

I. Kra は、タイヒミューラー空間論を用いて、$\mathcal{S}_1(\hat{F})$ の元の分類を与えた [10]。同様の手法で、今吉一伊藤一山本は、$\partial D_i$ $(i=1,\dots,n)$ を動かさないような $\mathcal{S}_n(\hat{F})$ の元の分類を与えた [8] ($n=2$ の場合)、[9] ($n \geq 2$ の場合)。

1.3. **結果.** 本稿の目的は、$\mathcal{S}_n(\hat{F})$ の全ての元のニールセン・サーストン型の分類を、位相幾何的な手法で与えることである。主結果を述べる為に、いくつか用語を準備する。

**Definition 1** ($f$ に付随する組み紐). $f$ を $\mathcal{S}_n(F)$ の元とし、$\Phi$ を $f$ から恒等写像までのイソトピーとする: つまり、$\Phi$ を $F \times I \to F \times I$ の同相写像で、$\Phi(x,0)=(f(x),0)$ かつ $\Phi(x,1)=(x,1)$ を満たすものとする。このとき、$t_i^f : I \to F \times I$ を $t_i^f(t) = \Phi(x_j,t)$ で定義する (ただし、$x_i = f(x_j)$)。このとき、$t_1^f,\dots,t_n^f$ により、$F \times I$ 上の組み紐 $b^f = (t_1^f(I),\dots,t_n^f(I);\ F \times I)$ が定まる。これを **$f$ に付随する組み紐**(braid associated to $f$) ということにする。得られた $(f(x_j),0)=(x_i,0)$ と $(x_j,1)$ を結ぶ単調な弧を、$i$ **番目の弦**($i$-th string) ということにする。各弦 $t_i^f$ には、$t_i^f(0)=(x_i,0)$ から $t_i^f(1)=(x_j,1)$ にむけて向きをつけておく。[1]

$F \times I$ の組み紐で、各 $i$ 番目の弦が $(x_i,0)$ と $(x_j,1)$ とを結ぶ単調な弧であるものの集合を $\mathrm{Br}_n(F)$ と表すことにする。$\mathrm{Br}_n(F)$ の二つの組み紐 $b$, $b'$ に対し、$F \times I$ の積構造を保ち、恒等写像にイソトピックで、$F \times \{0,1\}$ に制限すると恒等写像な微分同相写像 $G$ が存在して、$G(b)=b'$ となっているとき、$b$ と $b'$ は**同値**(equivalent) であるという。

$\mathcal{S}_n(\hat{F})$ の各同型写像 $\hat{f}$ に対して、対応する $f \in \mathcal{S}_n(F)$ は、Definition 1 のように組み紐 $b^f \in \mathrm{Br}_n(F)$ を定める。逆に、各組み紐 $b \in \mathrm{Br}_n(F)$ に対し、$b^f = b$ となる $f \in \mathcal{S}_n(F)$ が存在して、当然、その制限 $\hat{f}$ は $\mathcal{S}_n(\hat{F})$ の元となる。実際、次のような 1 対 1 対応が知られている。

**Proposition 1.1.** $\hat{f}$ と $b^f$ を対応させるという写像 $\Psi : \mathcal{S}_n(\hat{F}) \to \mathrm{Br}_n(F)$ は、自然な同型写像 $\bar{\Psi}$ : $\mathcal{S}_n(\hat{F})/isotopy \to \mathrm{Br}_n(F)/equivalence$ を与える。特に、組み紐 $b^f$ の同値類は、$F$ から恒等写像へのイソトピー $\Phi$ の取り方に依らず定義される。

さらに、組み紐に関して以下の用語を準備する。

**Definition 2.** $b = (t_1,\dots,t_n;\ F \times I)$ を $\mathrm{Br}_n(F)$ の組み紐とする。また、$b$ の各弦 $t_i$ は、$F \times \{0\} \cap t_i$ から $F \times \{1\} \cap t_i$ に向けて、向きづけられているとする。

(1) 組み紐 $(\{x_1\} \times I,\dots,\{x_n\} \times I;\ F \times I)$ と同値であるとき、**自明**(trivial) であるという。

(2) $F \times I$ から $F$ への自然な射影を $p : F \times I \to F$ であらわすことにする。$b$ の部分族 $\{t_{i_1},\dots,t_{i_k}\}$ が、条件 $p((t_{i_1} \cup \cdots \cup t_{i_k}) \cap (F \times \{0\})) = p((t_{i_1} \cup \cdots \cup t_{i_k}) \cap (F \times \{1\}))$ を満たし、そのどの真部分族も

[1] 以下では、記号を濫用して、写像とその像をしばしば同じ記号で表す。例えば、$t_i^f$ で $t_i^f$ の像 $t_i^f(I)$ も表すことにする。

この条件を満たさないとき、**巡回族**(cyclic family) であるということにする。各巡回族 $\{t_{i_1},\ldots,t_{i_k}\}$ に対し、$p(t_{i_1}),\ldots,p(t_{i_k})$ を $t_{i_1},\ldots,t_{i_k}$ の向きから誘導された向き付きの弧とみなし、(適当な順序で) 向きづけられた弧としての積をとることによって、$F$ 上の閉曲線が定義される。これを $c_j$ と表すことにする。

(3) $b_1,\ldots,b_m$ を $f$ に付随する組み紐 $b$ の巡回族とし、$c_1,\ldots,c_m$ を対応する $F$ 上の閉曲線とする。このとき、$\mathcal{C}=\{c_1,\ldots,c_m\}$ を、**$b$ に付随する閉曲線族**(a system of closed curves associated to $b$) ということにする。

(4) 組み紐 $b$ に対し、対応する閉曲線族 $\mathcal{C}=\{c_1,\ldots,c_m\}$ が $F$ 上で充填的 (filling) である (つまり、$c_1\cup\cdots\cup c_m$ が $F$ 上の任意の非自明な閉曲線と交わる) とき、$b$ は**充填的**(filling) であるという。さらに、$b$ と同値な全ての組み紐が充填的であるとき、$b$ は**安定充填的**(stably filling) ということにする。

(5) $b$ の部分集合 $\{t_{i_1},\ldots,t_{i_k}\}$ $(k\geq 2)$ に対し、積構造を保つ埋め込み $\eta: D_1^2\times I\cup\cdots\cup D_{k'}^2\times I\to F\times I$ で、その像が $t_{i_1},\ldots,t_{i_k}$ を含み、他の弦と交わらず、$p(\eta(D_1^2\times\{0\}\cup\cdots\cup D_{k'}^2\times\{0\}))=p(\eta(D_1^2\times\{1\}\cup\cdots\cup D_{k'}^2\times\{1\}))$ を満たすものが存在するとき、**平行族**(parallel family) であるということにする。Figure 1 (1) を参照。

(6) $b$ の部分集合 $\{t_{i_1},\ldots,t_{i_k}\}$ $(k\geq 1)$ に対し、$F\times I$ の境界 $(\partial F)\times I$ の成分の近傍 $N$ で、$t_{i_1},\ldots,t_{i_k}$ を含み、他の弦と交わらず、$p(N\cap(F\times\{0\}))=p(N\cap(F\times\{1\}))$ を満たすものが存在するとき、**周辺族**(peripheral family) であるということにする。Figure 1 (2) を参照。

(7) $b$ の平行族、または周辺族である部分集合を、**$P$-族**($P$-family) と呼ぶことにする。

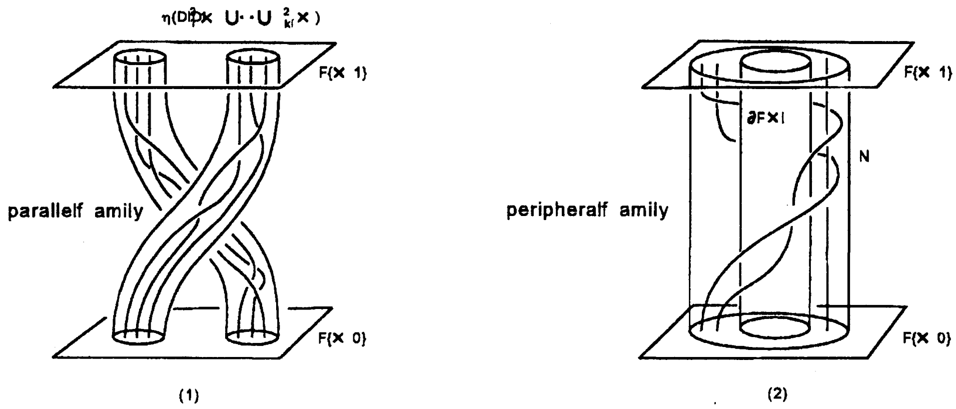


FIGURE 1. parallel family and peripheral family

以上で、主定理を述べる準備ができた。

**Theorem 1.2.** $f$ を $\mathcal{S}_n(F)$ の元である $F$ の同型写像とし、$b^f$ を付随する組み紐とする。

(i) $\hat{f}$ が周期的写像にイソトピックである必要十分条件は、$b^f$ が自明であることである。

(ii) $\hat{f}$ が可約写像にイソトピックである必要十分条件は、$b^f$ が $P$-族を含むか、または、安定充填的でないことである。

(iii) $\hat{f}$ が擬アノソフ写像にイソトピックである必要十分条件は、$b^f$ が安定充填的であり、かつ、$P$-族を含まないことである。

*Remark.* 実際、もし $b^f$ が自明ならば、Proposition 1.1 より、$\hat{f}$ は恒等写像にイソトピックである。従って、(i) が言っていることは、$\hat{f}$ が周期的写像にイソトピックである必要十分条件は、それが恒等写像にイソト

ピックであることとなる。また、安定充填的な組み紐は非自明でなければならないので、(i) と (ii) から、既約な (i.e., 可約になりえない)$\mathcal{S}(\hat{F})$ の元は、周期的写像にイソトピックにならないことがわかる。これにより、任意の $\mathcal{S}(\hat{F})$ の既約同型写像は、擬アノソフ写像にイソトピックとなり、結局、(ii) と (iii) は同値であることになる。

1.4. **系.** 2つの閉曲線族 $\mathcal{C} = \{c_1, \cdots, c_m\}$、$\mathcal{C}' = \{c'_1, \cdots, c'_m\}$ が**同値**(equivalent) であるとは、各 $c_i$ が (閉曲線として)$c'_i$ にホモトピックであるということである。

**Definition 3.** $\mathcal{C} = \{c_1, \ldots, c_m\}$ を $F$ 上の閉曲線族とする。

(1) $\mathcal{C}$ に対し、$\mathcal{C}$ と同値な任意の $\mathcal{C}'$ が充填的であるとき、**安定充填的**(stably filling) であるということにする。

(2) 安定充填的な閉曲線族 $\mathcal{C}$ が次の条件を満たすとき、**性質 (∗) を持つ**ということにする。

  (i) 任意の $c_i$ は**原始的**(primitive) (i.e., どんな閉曲線 $c$ をとってきても、$c_i$ は $c^p$ とホモトピックにならない　$(p \geq 2)$)。

  (ii) $c_i$ と $c_j$ は (閉曲線として) ホモトピックでない $(i \neq j)$。

  (iii) $c_i$ は境界 $\partial F$ にホモトープできない。

$f$ に付随する組み紐 $b^f$ に付随する閉曲線族を **$f$ に付随する閉曲線族**(a system of closed curves associated to $f$) と呼ぶことにし、$\mathcal{C}^f = \{c^f_1, \ldots, c^f_m\}$ で表す。

同型写像に付随する閉曲線族を用いると、次の系が得られる。

**Corollary 1.3.** $f$ を $\mathcal{S}_n(F)$ の元である同型写像とし、$\mathcal{C}^f = \{c^f_1, \ldots, c^f_m\}$ を付随する閉曲線族とする。もし、$\mathcal{C}^f$ が性質 (∗) を持つならば、$\hat{f}$ は擬アノソフ写像にイソトピックになる。

*Remark.* (*a*) もし $f$ が $x_1, \ldots, x_n$ を各点ごとに固定しているならば、性質 (∗) の条件 (i) は不要になる (Claim 5.1 の証明を参照)。とくに、$n = 1$ の場合、$b^f$ は1弦の組み紐になり平行族を含み得ないことから、性質 (∗) の定義は、「$\{c_1\}$ が安定充填的で、かつ、$c_1$ は境界 $\partial F$ にホモトープできない」と簡略化される。(*b*) Corollary 1.3 の逆は、$n \geq 2$ の一般の場合、成立しない。しかし、$n = 1$ の場合には、(*a*) の簡略化された性質 (∗) を用いれば、逆もまた正しくなる ([10] を参照)。

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## 2. EXAMPLE

In this section, we give some applications of Corollary 1.3.

**Example 1.** Let $f$ be an automorphism in $\mathcal{S}_3(F)$ such that $f(x_1) = x_2, f(x_2) = x_1, f(x_3) = x_3$ and $\mathcal{C}^f$ is given by Figure 2. Then $\hat{f}$ is isotopic to a pseudo-Anosov automorphism in $\mathcal{S}_3(\hat{F})$. Note that the automorphism $f$ with the the given system of closed curves $\mathcal{C}^f$ below is not unique, but for each $f$, $\hat{f}$ is isotopic to a pseudo-Anosov automorphism in $\mathcal{S}_3(\hat{F})$.

In fact, by Corollary 1.3, it is sufficient to show that $\mathcal{C}^f = \{c_1, c_2\}$ has property (∗), where $c_1 = p(t^f_1) * p(t^f_2)$ and $c_2 = p(t^f_3)$. It is straightforward to check that $\mathcal{C}^f$ satisfies (i), (ii) and (iii). To show that it is stably filling, we first find a hyperbolic structure on $F$ such that the curve $c_1$ is realized as a closed geodesic. In fact this can be done by decomposing $F$ into a pair of pants. Then it is known that a closed geodesic on a closed hyperbolic surface which cuts the surface into open disks is stably filling.

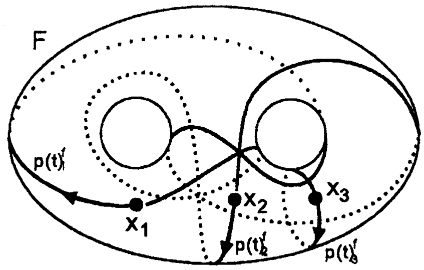



FIGURE 2

Since $F - c_1$ consists of open disks, $\{c_1\}$ is stably filling, and is also $\{c_1, c_2\}$. This fact can be also checked by [7], in which Hass and Scott gave a combinatorial criteria showing the given system of closed curves are stably filling.

**Example 2.** Let $f$ be an automorphism in $\mathcal{S}_3(F)$ such that $f(x_1) = x_3, f(x_2) = x_1, f(x_3) = x_2$ and $\mathcal{C}^f$ is given by Figure 3. Then the same argument as above shows that $\mathcal{C}^f = \{c_1\}$ $(c_1 = p(t_1^f) * p(t_2^f) * p(t_3^f))$ has property $(*)$ and $\hat{f}$ is isotopic to a pseudo-Anosov automorphism in $\mathcal{S}_3(\hat{F})$.

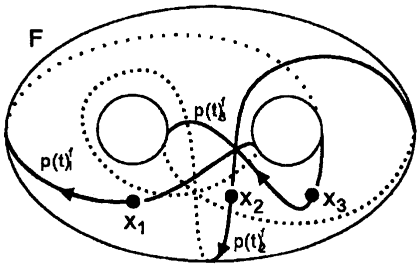



FIGURE 3

## 3. ISOTOPIES OF ESSENTIAL CIRCLES ON A SURFACE

In this section we will prove the following result which implies that an isotopy sending a family of circles on a surface $F$ to themselves is essentialy unique if $F$ has negative Euler characteristic.

Let $F$ be a compact, orientable surface of negative Euler characteristic and $a_1, \ldots, a_k$ mutually isotopic, pairwise disjoint essential circles on $F$. Let $A_1, \ldots, A_k$ be a pairwise disjoint, monotone (meaning no local maxima and minima) annuli in $F \times I$ such that $p(\partial(A_1 \cup \cdots \cup A_k)) = a_1 \cup \cdots \cup a_k$. Then a map $\sigma : \{1, \ldots, k\} \to \{1, \ldots, k\}$ is determined so that $A_i$ connects $a_{i,0} = a_i \times \{0\}$ and $a_{\sigma(i),1} = a_{\sigma(i)} \times \{1\}$. $A_1 \cup \cdots \cup A_k$ corresponds to an isotopy sending $a_1 \cup \cdots \cup a_k$ to itself. Then we have:

**Lemma 3.1.** (1) $\sigma(i) = i$ *for* $i = 1, \ldots, k$, *i.e.*, $\partial A_i = a_i \times \{0, 1\}$. (2) $A_i$ *can be isotoped to a vertical annulus* $a_i \times I$ *by a level preserving isotopy which is the identity on* $F \times \{0, 1\}$.

Proof. First we suppose that $F$ is a closed surface of genus $g$. Choose a family of $2g$ essential simple closed curves $\varepsilon_1, \cdots, \varepsilon_{2g}$ on $F$ as in Figure 4; $\cup_{k=1}^{2g} \varepsilon_k$ cuts $F$ into a single disk and $a_i$ is homologous to none of $\varepsilon_1, \cdots, \varepsilon_{2g}$. Without loss of generality, we may assume that the curve $a_i$ is precisely as in Figure 4 (1) or (2) depending on whether $a_i$ is non-separating or separating: $a_i \cap \varepsilon_4 = \{z_i\}$, $a_i \cap \varepsilon_j = \{z_i, z_i'\}$.

In fact, for a given essential simple loop $a_i$ on $F$, there is a diffeomorphism $h : F \to F$ sending $a_i$ to the curve as in Figure 4 (1) or (2). Then we have the required situation by applying $h\times$ id. $: F \times I \to F \times I$.


(1)ai ; sn on- separating (2)ai ; ss eparating

FIGURE 4

In the following we may relabel the indices and orient $a_i$ so that $a_1, \ldots, a_k$ are homotopic as oriented curves, and if $a_i$ is separating, then $a_i$ intersects $\varepsilon_j$ at $z_i$ and $z_i'$ with opposite directions and $(a_1 \cup \cdots \cup a_k) \cap \varepsilon_j$ appears $z_k, \ldots, z_2, z_1, z_1', z_2', \ldots, z_k'$ in circular ordering on $\varepsilon_j$.

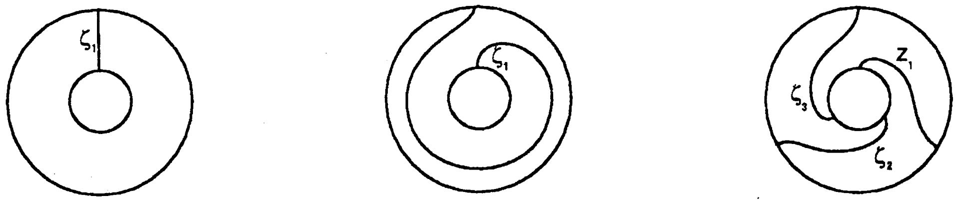
Let $E_k$ be the vertical annulus $p^{-1}(\varepsilon_k)$ for $1 \le k \le 2g$.

Since $a_i$ is essential, $A_i$ is incompressible. Thus we may assume, by a level preserving isotopy fixing $F \times \{0,1\}$, that each $A_i$ intersects $E_4$ (resp. $E_j$) transversely and that each component of $A_i \cap E_4$ (resp. $A_i \cap E_j$) does not bound a disk in $E_4$ (resp. $E_j$). Note that each level preserving isotopy keeps $A_i$ monotone.

We first consider the case where $a_i$ is non-separating.

**Claim 3.2.** *$A_i \cap E_4$ consists of an arc $\zeta_i$ isotopic to a vertical segment by a level preserving isotopy leaving its boundary invariant.*

*Proof.* Since $a_i \cap \varepsilon_4 = \{z_i\}$, there is no boundary-parallel arc in $E_4$. Hence $(A_1 \cup \cdots \cup A_k) \cap E_4$ consists of an essential monotone arc, say as in Figure 5.




k=1 k=1 k=3

FIGURE 5

Take a subfamily $A_1, A_{\sigma(1)}, \ldots, A_{\sigma^{j-1}(1)}$ of the annuli such that $j \in \{1, \ldots, k\}$ satisfies $\sigma^j(1) = 1$ and no proper subfamily satisfies this property. Let $T'$ be a torus obtained from $A_1 \cup A_{\sigma(1)} \cup \cdots \cup A_{\sigma^{j-1}(1)}$ by identifying their boundaries via the identification $(x, 0) = (x, 1)$. Then there is a map $p'$ such that the following diagram commutes:

Connecting the arcs $\zeta_1, \zeta_{\sigma(1)}, \ldots, \zeta_{\sigma^{j-1}(1)}$ in a suitable order, we obtain an essential loop $\alpha$ on $T'$, which satisfies $p'_*([\alpha]) = [\varepsilon_4]^m \in \pi_1(F, z_1)$ for some integer $m$. Assume for a contradiction that $m > 0$. Then $p'_*([\alpha])$ is nontrivial. The essential loop $a_{1,0}$ also gives an essential loop $\beta$ on $T'$. Note that since $[\alpha][\beta] = [\beta][\alpha] \in \pi_1(T')$, $p'_*([\alpha])p'_*([\beta]) = p'_*([\beta])p'_*([\alpha])$ in $\pi_1(F, z_1)$. Furthermore, since $|\varepsilon_4 \cap a_1| = 1$,

$$\begin{tikzcd}
A_1 \cup A_{\sigma(1)} \cup \cdots \cup A_{\sigma^{j-1}(1)} \arrow[r] \arrow[d, "p"'] & T' \arrow[dl, "p'"] \\
F &
\end{tikzcd}$$

$p'_*([\alpha])$ and $p'_*([\beta])$ generate a rank two free abelian subgroup in $\pi_1(F, z_1)$. This contradicts that the genus of $F$ is greater than one.

It follows that $m = 0$, hence $p(\partial\zeta_i) = z_i$ and we can isotope $A_i$ (fixing $F \times \{0,1\}$) so that $A_i \cap E_4$ consists of a single vertical segment. □(Claim 3.2)

This claim implies the first assertion of Lemma 3.1 in the case where $a_i$ is non-separating.

Now let us show that $A_i$ can be isotoped to the vertical annulus as required. Since $E_3 \cap E_4$, $E_4 \cap E_5$ (if $g > 2$) and $A_i \cap E_4$ consist of a vertical segment respectively, we can isotope without changing $A_i \cap E_4$ so that $A_i \cap E_3 = \emptyset$ and $A_i \cap E_5 = \emptyset$ (if $g > 2$); here we use also a fact that $a_i \cap \varepsilon_3 = \emptyset$, $a_i \cap \varepsilon_5 = \emptyset$ (if $g > 2$) and an incompressibility of $A_i$. For other $E_s$ ($s \neq 3, 4, 5$), since $a_i \cap \varepsilon_s = \emptyset$ and $A_i$ is incompressible, we can isotope further by a level preserving isotopy fixing $F \times \{0,1\}$ so that $A_i \cap E_s$ is empty or consists of essential circles in $E_s$; each circle is also essential in $A_i$ because $E_s$ is incompressible. In the latter case $a_i$ is homotopic to $\varepsilon_s$, a contradiction. Since $A_i$ intersects only $E_4$ or $E_j$ in vertical segments and $E_1 \cup \cdots \cup E_{2g}$ cuts $F \times I$ into a [disk]$\times I$, we can isotope $A_i$ to the vertical annulus by a level preserving isotopy as desired.

Next we consider the case where $a_i$ is separating.

In this case, $A_i \cap E_j$ consists of two properly embedded arcs $\zeta_i$ and $\zeta'_i$ in $E_j$.

**Claim 3.3.** *$\partial\zeta_i = \{(z_i, 0), (z_i, 1)\}$, and hence $\partial\zeta'_i = \{(z'_i, 0), (z'_i, 1)\}$.*

*Proof.* If $\zeta_i$ is boundary-parallel arc, then since $A_i$ is boundary-incompressible and $F \times \{0\}$ is incompressible, there should be a bigon $D \subset F$ with $\partial D = d_1 \cup d_2$ such that $d_1 \subset a_i$ and $d_2 \subset \varepsilon_j$. This is impossible, see Figure 4 (2). Thus $\partial\zeta_i = \{(z_i, 0), (z_i, 1)\}$ or $\partial\zeta_i = \{(z_i, 0), (z'_i, 1)\}$, for otherwise, as in Figure 6 (1), there would be a boundary-parallel arc in $(A_1 \cup \cdots \cup A_k) \cap E_j$. In fact, since no $\zeta_i$'s are boundary parallel, these arcs define a bijection $\tau$ on $\{z_k, \ldots, z_1, z'_1, \ldots, z'_k\}$ so that $\zeta_i$ connets the points $(z_i, 0)$ and $(\tau(z_i), 1)$. Then since $\zeta'_i$ is also a component of $A_i \cap E_j$, $\zeta'_i$ connets the points $(z'_i, 0)$ and $(\tau(z'_i), 1) = (\tau(z_i)', 1)$. If $\tau(z_i) = z_j$ (resp. $\tau(z_i) = z'_j$), then $\tau(z_i)'$ denotes $z'_j$ (resp. $z_j$).

Let us show that $\tau(z_i) = z_i$. Suppose to the contrary that $\tau(z_i) \neq z_i$ for some $i$. If $\tau(z_i) = z_j$ for some $j \neq i$, say as in Figure 6 (1) in which $i = 1$ and $j = 2$, then there would be a boundary-parallel arc in $(A_1 \cup \cdots \cup A_k) \cap E_j$, a contradiction. If $\tau(z_i) = z'_j$ for some $j$, say asin Figure 6 (2) in which $i = 2$ and $j = 2$, then sliding the oriented closed curve $a_{i,0}$ along the annulus $A_i$ to obtain an oriented closed curve $a_{j,1}$. Then since $a_{j,0}$ is orientedly homotopic to $a_{i,0}$, $p(a_{j,1})$ and $p(a_{j,0})$ have opposite orientations. This implies that $a_j$ and $\bar{a}_j$ (the closed curve obtained from $a_j$ by inverting its orientation) and freely homotpic in $F$, hence $F$ would be non-orientable, a contradiction.

□(Claim 3.3)

Thus we have a situation, say as in Figure 7.

This observation implies the first assertion of Lemma 3.1 in the case where $a_i$ is separating.

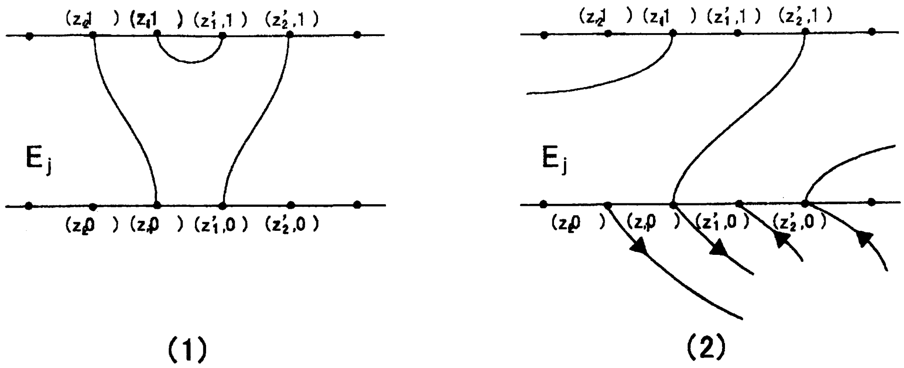



FIGURE 6

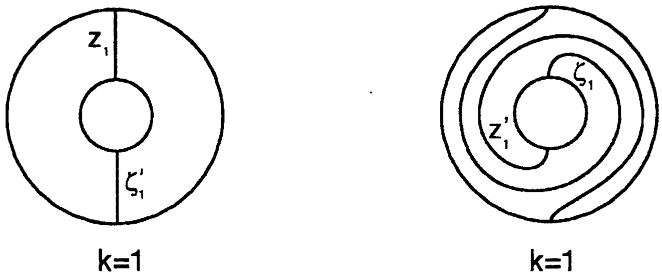



FIGURE 7

Let us show that $A_i$ is also isotoped to the vertical annulus as required in this case. By using the same argument in the proof of Claim 3.2 for a torus $T'$ obtained from single $A_i$, $\zeta_i$ and $\zeta_i'$ are shown to be isotopic to vertical segments by a level preserving isotopy leaving their boundaries invariant. Then, as in the above, we can isotope $A_i$ (fixing $F \times \{0,1\}$) so that $A \cap E_s = \emptyset$ $(s \neq j)$, thus we can isotope $A_i$ to the vertical annulus by a level preserving isotopy as desired.

Finally suppose that $F$ has genus $g$ and $d$ boundary components. We can find a system of properly embedded arcs $\{\varepsilon_1, \dots, \varepsilon_{2g}, \delta_1, \dots, \delta_{d-1}\}$ so that they cut $F$ into a single disk. Then the result follows by applying the same argument as above. (The proof is easier, because $p^{-1}(\varepsilon_j)$ and $p^{-1}(\delta_k)$ is a rectangle, not an annulus.) □(Lemma 3.1)

## 4. Proof of Theorem 1.2

Let $f$ be an element in $\mathcal{S}_n(F)$ and $b^f$ an associated braid.

4.1. **Proof of (i).** This is certainly well-known, but for completeness, we give a proof. If $b^f$ is trivial, then $\hat{f}$ is isotopic to the identity map, which has period 1. Conversely if $\hat{f}$ is isotopic to a periodic automorphism, then by Proposition 1.1, $b^f$ has a finite order in the braid group. If $b^f$ is nontrivial, then [5, Theorem 8] shows that $F$ would be $S^2$ or the projective plane $\mathbb{R}P^2$, contradicting our assumption. Thus $b^f$ is trivial.

4.2. **Proof of the "only if" part of (ii).** Assume that $\hat{f}$ is isotopic to a reducible automorphism. Then there is an essential 1-submanifold $C = a_1 \cup \cdots \cup a_m \subset F$ such that $f(C)$ is isotopic to $C$ on $\hat{F}$. In the following, we assume that $f(a_{k_i})$ is isotopic to $a_i$, i.e., $a_{k_i}$ is isotopic to $f^{-1}(a_i)$ $(i = 1, \dots, m)$ on $\hat{F}$.

The isotopy from $f^{-1}(a_i)$ to $a_{k_i}$ $(1 \le i \le m)$ on $\hat{F}$ is realized as a family of monotone annuli $\tilde{A}_1, \dots \tilde{A}_m$ in $\hat{F} \times I \subset F \times I$ so that $\partial \tilde{A}_1 = (f^{-1}(a_1) \times \{0\}) \cup (a_{k_1} \times \{1\}), \dots, \partial \tilde{A}_m = (f^{-1}(a_m) \times \{0\}) \cup (a_{k_m} \times \{1\})$. Note that $\tilde{A}_i \cap (\{x_j\} \times I) = \emptyset$ for $i = 1, \dots, m$, $j = 1, \dots, n$. Since $f$ is isotopic to the identity on $F$, we have a level preserving diffeomorphism of $F \times I$ sending $(x, 0)$ to $(f(x), 0)$ and $(x, 1)$ to $(x, 1)$, which deforms also the vertical segment $\{x_j\} \times I$ to a monotone arc $t_i^f$ with $\partial t_i^f = \{(x_i, 0), (x_j, 1)\}$, where $x_i = f(x_j)$. Then $t_1^f, \dots, t_n^f$ define a braid $b^f$ in $F \times I$ (see, Definition 1). Simultaneously, the annuli $\tilde{A}_1, \dots, \tilde{A}_m$ are also deformed to a family of monotone annuli $A_1, \dots, A_m$ in $F \times I$, each of which is disjoint from the braid $b^f$ and satisfies that $\partial A_i = (a_i \times \{0\}) \cup (a_{k_i} \times \{1\})$. Let us choose annuli $A_1, \dots, A_k$ (after changing their indices if necessary) so that $p(\partial(A_1 \cup \dots \cup A_k) \cap (F \times \{0\})) = p(\partial(A_1 \cup \dots \cup A_k) \cap (F \times \{1\}))$ and no proper subset satisfy this property.

If $a_i$ bounds a disk $D_i$ on $F$, then since $C$ is an essential 1-submanifold on $\hat{F}$, $D_i$ contains at least two points of $\{x_1, \dots, x_n\}$. Then for each $i$, $\partial A_i \cap (F \times \{0\})$ bounds a disk $D_{i,0} \subset F \times \{0\}$ and $\partial A_i \cap (F \times \{1\})$ bounds a disk $D_{i,1} \subset F \times \{1\}$. By the irreducibility of $F \times I$, the 2-sphere $A_i \cup D_{i,0} \cup D_{i,1}$ bounds a 3-ball $B_i$. It turns out that each $B_i$ contains $m$ strings in $b^f$ for some integer $m \ge 2$ independent of $i$. The collection of strings in $b^f$ each of which is contained in $B_1 \cup \dots \cup B_k$ would be a parallel family, contradicting the assumption.

Hence $a_i$ is also essential on $F$ and $A_i$ is incompressible in $F \times I$. Then Lemma 3.1 (1) implies that $k = 1$ and Lemma 3.1 (2) shows that $A_1$ can be isotoped to the vertical annulus $a_1 \times I$ by a level preserving isotopy fixing $F \times \{0, 1\}$. Under this level preserving isotopy, the braid $b^f = (t_1^f, \dots, t_n^f;\ F \times I)$ is also isotoped to another braid $b' = (t_1', \dots, t_n';\ F \times I)$ (without moving their endpoints), which is equivalent to $b^f$; they define equivalent systems of closed curves. Since the annulus $A_1$ is disjoint from $b^f$, $a_1 \times I$ does not intersect $b'$ neither, and hence $a_1 \cap (\cup_{i=1}^n p(t_i')) = p(a_i \times I) \cap p(b') = p((a_i \times I) \cap b') = p(\emptyset) = \emptyset$. Since $b^f$ is stably filling, by definition, $\cup_{i=1}^n p(t_i')$ must intersect every essential embedded loop. It follows that $a_1$ would be parallel to a component of $\partial F$ with the parallelism containing some specified points $x_i$. This implies that $a_1 \times I$, and hence $A_1$, is the frontier of a collar neighborhood $N(\cong S^1 \times I \times I)$ of a component of $(\partial F) \times I$. Since $N(\cong S^1 \times I \times I)$ contains some strings $t_i^f$, there would be a peripheral family each member of which is in $N$. This contradicts $b^f$ having no $P$-families.

4.3. **Proof of the " if' part of (ii).** Suppose that the braid $b^f$ has a $P$-family or is not stably filling. Then by definition, (1) $b^f$ has a parallel family $\{t_{i_1}^f, \dots, t_{i_k}^f\}$ or (2) $b^f$ has a peripheral family $\{t_{i_1}^f, \dots, t_{i_k}^f\}$, or (3) $b^f$ is equivalent to a braid $b' = (t_1', \dots, t_n';\ F \times I)$ such that $p(t_1') \cup \dots \cup p(t_n')$ does not intersect an essential embedded loop $a$.

In each case, $C = p(\eta(\partial(D_1^2 \times \{0\} \cup \dots \cup D_m^2 \times \{0\})))$, the frontier of $p(N \cap (F \times \{0\}))$ in $F$, or the embedded loop $a$ is an essential 1-submanifold which is isotopic to the image of $f$ on $\hat{F}$. This means that $\hat{f}$ is isotopic to a reducible automorphism.

## 5. Proof of Corollary 1.3

Corollary 1.3 follows immediately from Theorem 1.2 (iii) and the claim below.

**Claim 5.1.** *If the system of closed curves $C^f = \{c_1^f, \dots, c_m^f\}$ has property $(*)$, then the braid $b^f = (t_1^f, \dots, t_n^f;\ F \times I)$ is stably filling and has no P-families.*

*Proof.* Suppose that we have a parallel family $\{t_{i_1}, \dots, t_{i_k}\}$ $(k \ge 2)$, which consists of some cyclic subfamilies. The cyclic families give a subsystem of closed curves in $C^f$. Since $k \ge 2$, the subsystem contains a closed curve homotopic to a nontrivial power of a closed curve or a pair of mutually homotopic closed curves. (If $f$ fixes $x_1, \ldots, x_n$ pointwisely, i.e., $b^f$ is a pure braid, then we have the latter possibility.) This contradicts the assumption. If we have a peripheral family $\{t_{i_1}, \ldots, t_{i_k}\}$, then clearly $c_i^f$ is homotoped into a component of $\partial F$, contradicting the assumption. Let $b' = (t'_1, \ldots, t'_n;\ F \times I)$ be a braid equivalent to $b^f$. Then the system of closed curves $\mathcal{C}'$ corresponding to $b'$ is equivalent to $\mathcal{C}^f$. Since $\mathcal{C}^f$ is stably filling, by definition, $\mathcal{C}'$ is filling. □(Claim 5.1)

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